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A time scale-based analysis of the laminar convective phenomena

V.A.F. Costa

Departamento de Engenharia Mecânica, Universidade de Aveiro, Campus Universitário de Santiago, 3810.193 Aveiro, Portugal

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Abstract

Convective transport phenomena are usually analyzed starting from the mass, momentum and thermal energy differential conservation equations. These equations express balances involving different transport phenomena contributions (convection, diffusion and source). When worked with any available solutions' method, they are usually interpreted in this form. However, in a scale sense, they can be interpreted as relationships between time scales associated with each individual transport phenomenon. Once the individual time scales for convection, diffusion and source are defined, the common differential equations can be interpreted as algebraic relations between time scales. This time scale-based approach seems to be a very effective tool for problem analysis when applied to laminar boundary layer flows and to the Bénard convection problem. It leads to a unified, consistent and physically coherent interpretation of the governing dimensionless parameters obtained, as well as to a unified treatment of situations usually taken as unequal. This unified treatment also leads to a unified setting of transition to turbulence criteria even for very different physical situations.

Keywords: Scale analysis; Time scales; Laminar flow; Convection heat transfer; Dimensionless parameters

1. Introduction

Fundamental analytical studies of laminar convective phenomena can be made using available tools such as scale analysis, similarity analysis or integral analysis. Amoung these, the scale analysis is one of the most attractive to study boundary layer flows, as it is mainly the present case, due to its inherent simplicity and physical insight. In the work by Bejan [1], the method is extensively used with very good results. There, the mass conservation equation is interpreted as a mass balance, the momentum equations are interpreted as an inertial-viscous (and buoyancy, if it is the case) force balance and the thermal energy equation is interpreted as a convection–diffusion energy balance. The differential equations can be interpreted in this way, but they can also be interpreted as relationships between the different time scales present, associated with each particular transport phenomena involved.

This time scale-based treatment is a powerful tool that can be applied to convective boundary layer flows, the results obtained being expressed as momentum and thermal boundary layer thickness. Once the latter is known, the heat and momentum transfer parameters can be obtained easily, but this is beyond the scope of this work. The same boundary layer results can also be obtained by using the scale analysis in the classical form, i.e., applied over the original differential equations as expressing conservation principles. However, the time scale analysis has its own physical insight, leading to a unified and physically coherent time basis for problem analysis and definition of the dimensionless governing parameters, always obtained as time ratios. Interpreting the dimensionless parameters in this way, their (usual and uncontroversial) physical meaning can be judged, as well as their numerical values. The time scale-based approach here proposed also leads to an effective criterion to decide whether the external mixed convection is dominated either by forced or by natural convection, which differs from some of the well-established criteria. A unified setting of the transition to turbulence criteria for different physical situations is also obtained through this time scale-based approach. The onset of the convection in the Bénard convection problem is another situation that is analyzed applying this time scale based approach.

E-mail address: v_costa@mec.ua.pt (V.A.F. Costa).

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Nomenclature

Bo	Boussinesq number	
D	channel width	m
D	diffusion coefficient	$m^2 \cdot s^{-1}$
g	gravitational acceleration	$m \cdot s^{-2}$
Gr	Grashof number	
H	height	m
L	length	m
m	mass	kg
n	temperature difference factor	
P	pressure	$N \cdot m^{-2}$
Pe	Péclet number	
Pr	Prandtl number	
Ra	Rayleigh number	
Re	Reynolds number	
S	source term	
t	time	s
T	temperature	K
u, v	Cartesian velocity components	$m \cdot s^{-1}$
U, V	velocity	$m \cdot s^{-1}$
x, y	Cartesian co-ordinates	m

Greek symbols

α	thermal diffusivity	$m^2 \cdot s^{-1}$
β	volumetric expansion coefficient	K^{-1}
Δ	difference value; distance	
δ	boundary layer thickness	m
ν	kinematic viscosity	$m^2 \cdot s^{-1}$
ρ	density	$kg \cdot m^{-3}$
τ	time scale	
ϕ	generic intensive (specific) property	
Φ	generic extensive property related to ϕ	

Subscripts

b	buoyancy
c	convection
d	diffusion
FC	forced convection
max	maximum value
min	minimum value
NC	natural convection
u, v	referring to velocity or momentum
0	at the wall
0	at the centerline, in internal flow
ϕ	related with the ϕ property
∞	at the free-stream

2. Time scales

A particular transport phenomenon is characterized by its own time scale, obtained considering the simplest transient form of the differential equation involving the particular transport phenomenon under analysis.

In the search of a convective time scale, one writes

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0 \tag{1}$$

where ϕ is a generic specific variable. In a scale sense, if $\Delta t_c \sim (\tau_c)_L$, $u \sim V_c$, and $\Delta x \sim L$, where V_c is the velocity in the x direction, Eq. (1) leads to $\Delta\phi/(\tau_c)_L \sim V_c \Delta\phi/L$, the convective time scale $(\tau_c)_L$ being obtained as

$$(\tau_c)_L \sim \frac{L}{V_c} \tag{2}$$

The physical meaning of Eq. (2) is well-known: $(\tau_c)_L$ is the time needed for (any variable) to travel through the distance L with the velocity V_c. This convective time scale is the same for any particular variable ϕ under analysis, as the fluid velocity V_c is unique.

The time scale for diffusion is obtained starting from

$$\frac{\partial \phi}{\partial t} = D_\phi \frac{\partial^2 \phi}{\partial x^2} \tag{3}$$

where D_ϕ is the ϕ diffusion coefficient. In a scale sense, if $\Delta t_d \sim (\tau_{d,\phi})_L$ and $\Delta x \sim L$, the diffusive time scale $(\tau_{d,\phi})_L$ for the ϕ variable can be obtained as

$$(\tau_{d,\phi})_L \sim \frac{L^2}{D_\phi} \tag{4}$$

The physical meaning of Eq. (4) is the following: $(\tau_{d,\phi})_L$ is the time needed for the ϕ variable to *travel diffusively* through the distance L, when the diffusion coefficient is D_ϕ. Each particular ϕ is associated with its own diffusion coefficient D_ϕ, thus leading to a particular diffusive time scale for each particular meaning of ϕ. Comparing Eqs. (2) and (4), the *diffusion velocity* $V_{d,\phi}$ for the variable ϕ, through the length L, can be found as scaling as $V_{d,\phi} \sim D_\phi/L$, where it should be stressed that this diffusive velocity is length dependent.

The problems involving natural convection include a buoyancy source term in the vertical velocity momentum equation, corresponding a given time scale to the buoyancy phenomenon. This time scale is obtained from the simplest form of the vertical momentum equation,

$$\frac{\partial v}{\partial t} = g\beta\Delta T \tag{5}$$

using the Boussinesq hypothesis. In a scale sense, if $\Delta t_b \sim \tau_b$ and $\Delta v \sim V_b$, the thermal buoyancy time scale τ_b is obtained as

$$\tau_b \sim \frac{V_b}{g\beta\Delta T} \tag{6}$$

The physical meaning of this time scale is the following: τ_b is the time needed for the (modified gravitational) acceleration $g\beta\Delta T$ to act in such way as to induce a vertical velocity change $\Delta v \sim V_b$.

Once the three important time scales for this study are defined, the differential equations can be transformed and interpreted as establishing algebraic relationships between time scales.

3. Relationships between time scales

The usual two-dimensional steady mass, momentum and heat transfer phenomena are described by partial differential equations, that can be written in the general conservative form [2]

$$\underbrace{\frac{\partial}{\partial x}(\rho u\phi)}_{\text{convection},x} + \underbrace{\frac{\partial}{\partial y}(\rho v\phi)}_{\text{convection},y} = \underbrace{\frac{\partial}{\partial x}\left(\rho D_\phi \frac{\partial \phi}{\partial x}\right)}_{\text{diffusion},x} + \underbrace{\frac{\partial}{\partial y}\left(\rho D_\phi \frac{\partial \phi}{\partial y}\right)}_{\text{diffusion},y} + \underbrace{S_\phi}_{\text{source}} \tag{7}$$

The generic specific variable ϕ can stand for the unit (global mass conservation equation), u or v, temperature T or other.

Eq. (7) results form a differential control-volume balance involving the three transport phenomena contributions related with the (extensive) $\Phi = m\phi$ variable, by unit of volume and unit of time, that is, it is a balance involving terms of the type $\Delta\Phi/(\text{volume} \times \text{time})$. Assuming that the variation $\Delta\Phi$ and the volume under analysis are the same for all terms, assumptions usual when using scale analysis [1], a balance involving terms of the type (1/time) are obtained. If the different times (associated with the terms of different nature) needed to obtain this *same* change $\Delta\Phi$ over the same volume are retained, Eq. (7) can be rewritten as establishing the following relationship between the time scales:

$$\left(\tau_c^{-1}\right)_x, \left(\tau_c^{-1}\right)_y \sim \left(\tau_{d,\phi}^{-1}\right)_x, \left(\tau_{d,\phi}^{-1}\right)_y, \tau_{\text{source},\phi}^{-1} \tag{8}$$

where (a, b) means the greater value of a and b, in a scale sense. The first conclusion obtained from Eq. (8) is that fast phenomena are dominant, with associated short time scales.

4. Forced convection

4.1. Differential equations for the boundary layer

The boundary layer adjacent to the x-oriented flat plate represented in Fig. 1 is governed by the mass conservation equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9}$$

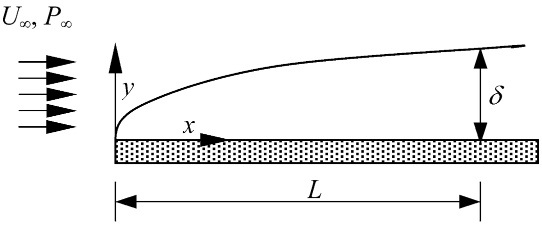

Fig. 1. Velocity boundary layer adjacent to a flat plate in forced flow.

and by the x and y momentum equations, joined together in the momentum boundary layer equation [1]

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\mathrm{d}P_\infty}{\mathrm{d}x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{10}$$

If thermal effects are present, the thermal energy boundary layer equation reads [1]

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{11}$$

4.2. Velocity boundary layer in external flow

Starting the analysis with Eq. (9), including only convective terms, the application of Eq. (8) leads to

$$\left(\tau_c^{-1}\right)_x \sim \left(\tau_c^{-1}\right)_y \tag{12}$$

that is, for two-dimensional flows the x and y convective time scales are of the same order. This result will be present in all the boundary layer flows analyzed in the present work. From Fig. 1, within the momentum boundary layer $x \sim L$, $y \sim \delta$, and $u \sim U_\infty$. Eq. (12) leads to $U_\infty/L \sim v/\delta$, that is,

$$v \sim \frac{\delta}{L}U_\infty \tag{13}$$

Assuming that $\mathrm{d}P_\infty/\mathrm{d}x = 0$ in Eq. (10), a usual simplifying assumption when using scale analysis [1], in a time scale sense one obtains

$$\left(\tau_c^{-1}\right)_x, \left(\tau_c^{-1}\right)_y \sim \left(\tau_{d,u}^{-1}\right)_y \tag{14}$$

As $x \sim L$, $y \sim \delta$, $u \sim U_\infty$ and $v \sim \delta U_\infty/L$, Eq. (14) leads to $(U_\infty/L)(1, 1) \sim \nu/\delta^2$, further algebra leading to

$$\frac{\delta}{L} \sim \left(\frac{\nu}{U_\infty L}\right)^{1/2} = Re_L^{-1/2} \tag{15}$$

where the L-based Reynolds number appears

$$Re_L \equiv \frac{U_\infty L}{\nu} \tag{16}$$

Usually, with $\mathrm{d}P_\infty/\mathrm{d}x = 0$, Eq. (10) is interpreted as (inertia forces) $\sim$ (viscous forces) [3], leading to

(inertia forces)/(viscous forces) ~ 1.

If the Reynolds number is interpreted as the inertial-viscous forces ratio, it should be always $Re_L \sim 1$. However, it is well known that the Reynolds number can reach values as high as 5×10^5 in external laminar boundary layer flows, a figure clearly different from 1. Bejan [1] points out such inconsistency in this Reynolds number interpretation, and proposes that the message of the Reynolds number is not given by the number itself but by its square root, given that $Re_L^{1/2} \sim L/\delta =$ slenderness ratio of the boundary layer.

In the present work, the physical meaning of the Reynolds number is given as a *time ratio*. Evaluating the following time ratio, as L is the known length,

$$\frac{(\tau_{d,u})_L}{(\tau_c)_L} \sim \frac{L^2/\nu}{L/U_\infty} = Re_L \tag{17}$$

one obtains the physical meaning of the Reynolds number itself: it is the ratio between the time needed for momentum diffusion through distance L and the time needed for convection through this *same* distance L, when the velocity is $u \sim U_\infty$. In most common situations, the distance L is such that leads to high values of the Reynolds number. However, the *real* diffusive process occurs through the $\delta(\delta \ll L)$ thickness only.

Taking the ratio

$$\frac{(\tau_{d,u})_\delta}{(\tau_c)_L} \sim \frac{\delta^2/\nu}{L/U_\infty} \sim 1 \tag{18}$$

one obtains the following time scale-based interpretation for the velocity boundary layer thickness: $\delta(L)$ is the distance, measured away from the solid wall, in such a way as to make the time needed for the momentum diffusion through distance $\delta(L)$ of the same order of the time needed for convection from 0 to L with velocity U_∞.

4.3. Velocity boundary layer in internal forced flow

The problem of internal forced flow differs from the external one essentially due to the presence of an upper horizontal wall, which precludes the flow assuming the free stream velocity. At the middle of the channel, and at a given length from the entry, the upper and lower boundary layers set linked, with a unique velocity value at the centerline. For a fully developed flow, the x momentum equation comes [1]

$$\nu \frac{d^2u}{dy^2} = \frac{1}{\rho}\frac{dP}{dx} \tag{19}$$

In a time scale sense, Eq. (19) informs us that $(\tau_{d,u}^{-1})_y \sim \tau_{\text{source},u}^{-1}$. In this case $y \sim D/2$, where D is the channel width, and $\tau_{\text{source},u} \sim u_0/[(1/\rho)(-dP/dx)]$, which is the time needed for the $(1/\rho)(-dP/dx)$ acceleration to act in order to induce the velocity change $\Delta u \sim u_0$ [the centerline velocity $u_0 = u(y = 0)$]. From $(\tau_{d,u}^{-1})_y \sim \tau_{\text{source},u}^{-1}$ the u_0 velocity scale is $u_0 \sim (D^2/4\nu)[(1/\rho)(-dP/dx)]$.

The exact solution of Eq. (19) can be obtained leading to the Hagen–Poiseuille solution for fully developed flow between parallel plates

$$u(y) = \frac{D^2}{8\nu}\left(-\frac{1}{\rho}\frac{dP}{dx}\right)\left[1 - \left(\frac{y}{D/2}\right)^2\right] \tag{20}$$

At the centerline

$$(y = 0), \quad u = u_0 = \left(D^2/8\nu\right)\left[(1/\rho)(-dP/dx)\right].$$

The best scale for $\tau_{\text{source},u}$ is obtained from the exact result for u_0 as

$$\tau_{\text{source},u} \sim D^2/8\nu \tag{21}$$

From the usual interpretation of the Reynolds number, as scaling with the ratio (inertia forces)/(viscous forces), the Reynolds number for fully developed flow between parallel plates should always be zero: there are no inertial forces in the x momentum equation (19). It is interesting to note that the Reynolds number, Re_D, defined using the average velocity $U = (D^2/12\mu)(-dP/dx)$, can reach values as high as 2300 in laminar regime. This usual interpretation erroneously suggests that the inertial effects are more important than the viscous ones in a Hagen–Poiseuille flow, which has no inertial effects.

The centerline velocity of an Hagen–Poiseuille flow is evaluated from Eq. (20) as $u_0 = (3/2)U$, and one can think of two boundary layers adjacent to the upper and lower solid walls, the (analogous of the) free stream velocity being u_0. Following the same procedure as for Eq. (17),

$$\frac{(\tau_{d,u})_D}{(\tau_c)_D} \sim \frac{D^2/\nu}{D/(3U/2)} \sim \frac{UD}{\nu} \equiv Re_D \tag{22}$$

is the Reynolds number, defined as a time ratio.

Very interesting is the analysis of the transition Reynolds number for these apparently very different situations, which is typically 5×10^5 for an external boundary layer flow and 2300 for an internal channel flow. Searching for the length scale, L, associated with the uniformly accelerated motion under the acceleration $(1/\rho)(-dP/dx)$, we have $L \sim (1/2)[(1/\rho)(-dP/dx)]\tau_{\text{source},u}^2$. Recalling Eqs. (20) and (21) one sets $L \sim u_0(D^2/16\nu)$. Evaluating the ratio similar to the one present in Eq. (17) we have

$$Re_L = \frac{u_0 L}{\nu} \sim \left(\frac{3}{8}\frac{UD}{\nu}\right)^2 = \left(\frac{3}{8}Re_D\right)^2 \tag{23}$$

The transition (D, U) based Reynolds number Re_D is typically 2300, the square value present in Eq. (23) (of the order of 7×10^5) being, in a scale sense, the transition (L, u_0) based Reynolds number for the same situation, taken as an external boundary layer situation, where u_0 is the counterpart of the free stream velocity. This aspect has been also pointed out by Burmeister [4], from the Reynolds number based on the momentum thickness.

4.4. Thermal boundary layer in external flow

The analysis starts considering the time scale ratio for heat and momentum diffusion through the *same* distance L,

$$\frac{(\tau_{d,T})_L}{(\tau_{d,u})_L} \sim \frac{L^2/\alpha}{L^2/\nu} = \frac{\nu}{\alpha} \equiv Pr \tag{24}$$

The Prandtl number—a fluid property—can be interpreted as the ratio between momentum and heat diffusive time scales associated with a common length.

The thermal energy conservation Equation (11) results in

$$\left(\tau_c^{-1}\right)_x, \left(\tau_c^{-1}\right)_y \sim \left(\tau_{d,T}^{-1}\right)_y \tag{25}$$

As, in the thermal boundary layer, $x \sim L$ and $y \sim \delta_T$, one obtains

$$\frac{u}{L}, \frac{v}{\delta_T} \sim \frac{\alpha}{\delta_T^2} \tag{26}$$

Taking the time ratio, similar to that of Eq. (18), assuming u/L as the dominant term in the left side of Eq. (26)

$$\frac{(\tau_{d,T})_{\delta_T}}{(\tau_c)_L} \sim \frac{\delta_T^2/\alpha}{L/u} \sim 1 \tag{27}$$

one obtains the following time scale-based interpretation for the thermal boundary layer thickness: $\delta_T(L)$ is the distance, measured away from the solid wall, in such a way as to make the time needed for heat diffusion through distance $\delta_T(L)$ of the same order of the time needed for convection from 0 to L, with the prevailing velocity u.

Thick thermal boundary layer ($\delta_T \gg \delta$). In this case as depicted in Fig. 2(a), the velocity at the exterior edge of the thermal boundary layer is $u \sim U_\infty$. The v velocity scale is obtained from Eq. (13), and Eq. (26) becomes $(U_\infty/L)(1, \delta/\delta_T) \sim \alpha/\delta_T^2$, leading to

$$\frac{\delta_T}{L} \sim \left(\frac{\alpha}{U_\infty L}\right)^{1/2} = (Re_L Pr)^{-1/2} = Pe_L^{-1/2} \tag{28}$$

Parameter

$$Pe_L \equiv \frac{U_\infty L}{\alpha} \tag{29}$$

is the Péclet number, similar to the Reynolds number of Eq. (16), and has the following time scale physical meaning: it is the ratio between the time needed for the heat diffusion through distance L and the time needed for the convection through this same distance L, with the velocity $u \sim U_\infty$. Thus, the Péclet number is for the forced external convective heat transfer (for a $Pr \ll 1$ fluid, as it will be seen further) what the Reynolds number is for the external convective momentum transfer.

From Eqs. (15) and (28) one obtains that $\delta_T/\delta \sim Pr^{-1/2} \gg 1$, that is, $\delta_T \gg \delta$ for a fluid with $Pr \ll 1$. Taking the time ratio

$$\frac{(\tau_{d,T})_{\delta_T}}{(\tau_{d,v})_\delta} \sim \frac{\delta_T^2/\alpha}{\delta^2/\nu} \sim 1 \tag{30}$$

one concludes that the diffusive times are similar. If the diffusive times are of the same order and $\delta_T \gg \delta$, it should be $\alpha \gg \nu$, that is, $Pr \ll 1$. It should be noted that $v/\delta_T \sim (U_\infty/L)(\delta/\delta_T) \ll U_\infty/L$, and it is correct to take u/L as the dominant term in the left side of Eq. (26) to obtain Eq. (27).

Thin thermal boundary layer ($\delta_T \ll \delta$). In this case, as shown in Fig. 2(b), the velocity at the exterior edge of the thermal boundary layer is, from geometrical considerations, $u \sim (\delta_T/\delta)U_\infty$. Within the thermal boundary layer, $x \sim L$, $y \sim \delta_T$ and $u \sim (\delta_T/\delta)U_\infty$, Eq. (12) leading to $v \sim (\delta_T/L)(\delta_T/\delta)U_\infty$. Application of the foregoing scales in Eq. (26) leads to $[(\delta_T/\delta)U_\infty/L](1, 1) \sim \alpha/\delta_T^2$, which gives $\delta_T/L \sim [(\delta/L)(\alpha/U_\infty L)]^{1/3}$. The δ/L ratio is obtained from Eq. (15), and one obtains that

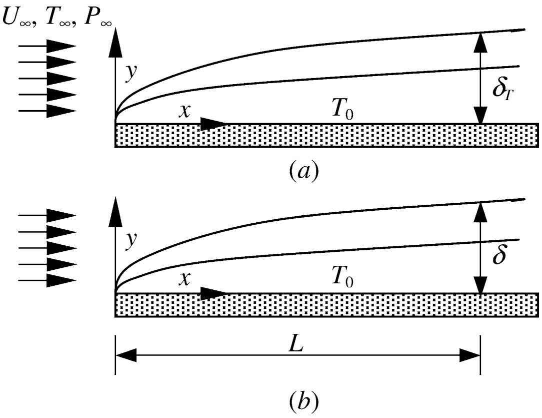

Fig. 2. Velocity and thermal boundary layers adjacent to a flat plate in forced flow: (a) Thick thermal boundary layer; (b) Thin thermal boundary layer.

$$\frac{\delta_T}{L} \sim Pe_L^{-1/2} Pr^{1/6} \tag{31}$$

From Eqs. (15) and (31) one obtains that $\delta_T/\delta \sim Pr^{-1/3} \ll 1$, that is, $\delta_T \ll \delta$ for a fluid with $Pr \gg 1$. The analogous of Eq. (30) gives $(\tau_{d,T})_{\delta_T}/(\tau_{d,T})_\delta \sim Pr^{1/3}$. If the diffusion times are of the order of $Pr^{1/3}$ and $\delta_T \ll \delta$, it should be $\alpha \ll \nu$, that is, $Pr \gg 1$. The left side of Eq. (26) becomes $(\delta_T/\delta)(U_\infty/L)$, $(\delta_T/L)(\delta_T/\delta)(U_\infty/\delta_T)$, both terms are of the same order, and it is correct to take u/L as the dominant term in the left side of Eq. (26) to obtain Eq. (27) even for $Pr \gg 1$.

5. External natural convection

In this case, the pressure-gradient term is balanced by the hydrostatic pressure term, and the momentum equation in the boundary layer adjacent to the vertical flat plate of Fig. 3 becomes [1]

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial x^2} + g\beta\Delta T \tag{32}$$

In what follows, $\Delta T = (T_{\max} - T_{\min})$. The thermal energy conservation equation for this case becomes

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial x^2} \tag{33}$$

5.1. Thermal boundary layer

When the thermal boundary layer represented in Fig. 3 is under analysis, $x \sim \delta_T$ and $y \sim H$, and Eq. (33) gives

$$\frac{u}{\delta_T}, \frac{v}{H} \sim \frac{\alpha}{\delta_T^2} \tag{34}$$

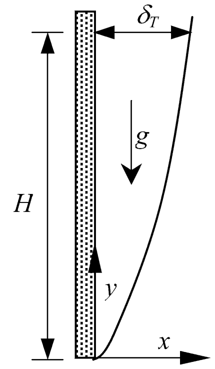

Fig. 3. Thermal boundary layer in external natural convection along an heated vertical wall.

From the time scale version of the mass conservation equation one obtains that $u/\delta_T \sim v/H$, and Eq. (34) can be written as

$$\frac{\delta_T}{H} \sim \left(\frac{\alpha}{vH}\right)^{1/2} \sim (Pe_{(v,H)})^{-1/2} \tag{35}$$

$Pe_{(v,H)}$ being the (v, H) based Péclet number. From Eq. (34) one can obtain that, within the thermal boundary layer

$$v \sim \frac{\alpha}{H}\left(\frac{H}{\delta_T}\right)^2 \tag{36}$$

Eq. (32) gives, within the thermal boundary layer:

$$\frac{u}{\delta_T}, \frac{v}{H} \sim \frac{\nu}{\delta_T^2}, \frac{g\beta\Delta T}{v} \tag{37}$$

Using the $u/\delta_T \sim v/H$ time scale version of the mass conservation equation, the v velocity scale within the thermal boundary layer can be obtained as

$$v \sim \frac{\alpha}{H}\left[Ra_H\left(Pr^{-1}, Pr^{-1}, 1\right)^{-1}\right]^{1/2} \tag{38}$$

where Ra_H is the H-based Rayleigh number defined as

$$Ra_H \equiv \frac{g\beta\Delta T H^3}{\nu\alpha} \tag{39}$$

If $Pr \gg 1$, Eqs. (38) and (36) give

$$v \sim \frac{\alpha}{H} Ra_H^{1/2} \tag{40}$$

$$\frac{\delta_T}{H} \sim Ra_H^{-1/4} \tag{41}$$

If $Pr \ll 1$, Eqs. (38) and (36) give

$$v \sim \frac{\alpha}{H} Bo_H^{1/2} \tag{42}$$

$$\frac{\delta_T}{H} \sim Bo_H^{-1/4} \tag{43}$$

The H-based Boussinesq number, Bo_H, is defined as

$$Bo_H \equiv Ra_H Pr \tag{44}$$

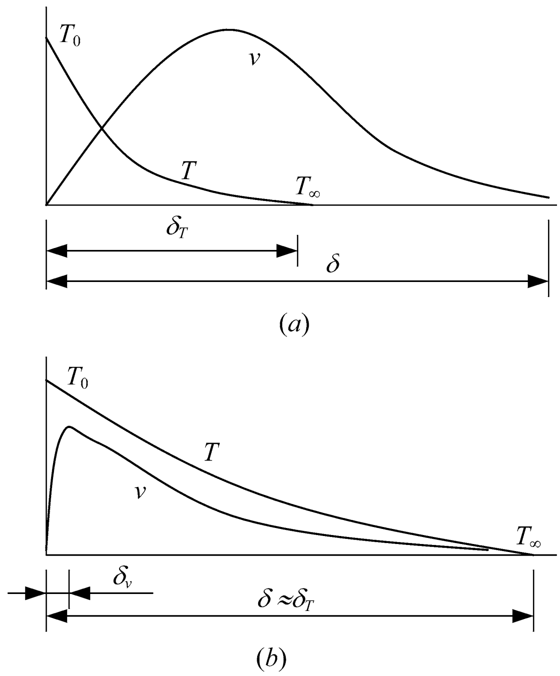

Fig. 4. Temperature and velocity profiles in external natural convection along an heated vertical wall for: (a) $Pr \gg 1$; (b) $Pr \ll 1$.

5.2. Velocity boundary layer

If $Pr \gg 1$, one would expect from the $\delta_T/\delta \sim Pr^{-1/3} \ll 1$ result, obtained for the forced convection case, that $\delta_T \ll \delta$ even for external natural convection, as illustrated in Fig. 4(a). In this case, the unheated flow (thus with no buoyancy effects) is viscously dragged by the moving heated layer of thickness δ_T. As the major portion of layer δ is not affected by buoyancy, the time scale relationship corresponding to Eq. (37), without the last right-hand side time scale and with δ replacing δ_T becomes

$$\frac{u}{\delta}, \frac{v}{H} \sim \frac{\nu}{\delta^2} \tag{45}$$

Remembering the v velocity scale given by Eq. (40), the mass conservation equation relationship $u/\delta_T \sim v/H$ and Eq. (41), one sets

$$\frac{\delta}{H} \sim Ra_H^{-1/4} Pr^{1/2} \tag{46}$$

If $Pr \ll 1$, the layer δ_T is heated, all the fluid within this layer moves up under the thermal buoyancy effect, and the thermal and velocity boundary layers would have the same thickness, situation illustrated in Fig. 4(b). Within the shear layer of thickness δ_v adjacent to the wall, the buoyancy and viscous effects are dominant. The time scale relationship corresponding to Eq. (37), without the convective time scales and with δ_v replacing δ_T becomes

$$\frac{\nu}{\delta_v^2} \sim \frac{g\beta\Delta T}{v} \tag{47}$$

The buoyancy velocity scale v is obtained from Eq. (42), and the appropriate δ_T/H ratio is obtained from Eq. (44), leading to

$$\frac{\delta_v}{H} \sim Bo_H^{-1/4} Pr^{1/2} \tag{48}$$

When comparing Eqs. (46) and (48), the different physical meaning of δ and δ_v should be retained, even if these equations are formally similar.

5.3. Dimensionless parameters

From the foregoing analysis, summarized by the results given by Eqs. (41), (43), (46) and (48), one concludes that the role played by the Rayleigh number Ra_H when $Pr \gg 1$ is essentially played by the Boussinesq number Bo_H when $Pr \ll 1$.

The usually ambiguous physical meaning of the H-based Grashof number, defined as

$$Gr_H \equiv \frac{g\beta \Delta T H^3}{\nu^2} \tag{49}$$

is analyzed first. The Grashof number is usually used *alone* as the natural convection governing parameter for natural convection heat and fluid flow problems. However, it is not the most suitable parameter to characterize all the natural convection situations, as given by Eqs. (41), (43), (46) and (48). This parameter is usually obtained from the nondimensionalized momentum only, which does not consider the correct vertical velocity scale within the thermal boundary layer, as given by Eq. (36). The usual interpretation of the Grashof number is [5] $Gr_H \sim$ (buoyancyforces)/(viscous forces). However, in terms of forces, Eq. (32) gives (inertial forces) $\sim$ (viscous forces, buoyancy forces) and, once the velocity scale given by Eq. (36) and the $u/\delta_T \sim v/H$ mass conservation equation relationship is introduced, the foregoing relation can be rewritten as (viscous forces) $\times\ (1, Pr^{-1}) \sim$ (buoyancy forces). For a $Pr \approx 1$ fluid, it should be $Gr_H \approx Ra_H \approx Bo_H \sim 1$, but we have laminar situations with Grashof numbers as high as 10^9, which is a measure of as how inadequate the usual physical meaning of this parameter is. Eq. (42) can be rewritten as $v \sim (\nu/H)Gr_H^{1/2}$, and Eq. (48) can be rewritten as $(\delta_v/H) \sim Gr_H^{-1/4}$, that is, there are some characteristics of the natural convection fluid flow problem that are properly described by the Grashof number. Bejan [1] gives an interpretation, not for the parameters Ra_H, Bo_H and Gr_H but, with the results of Eqs. (41), (43) and (48) present, for their 1/4th power, which gives the slenderness of the boundary layer region corresponding to the buoyancy induced flow. A possible interpretation of the 1/4th power of Ra_H and Bo_H is thus the ratio of the wall height to the thermal boundary layer thickness for $Pr \gg 1$ and $Pr \ll 1$, respectively. Similarly, the 1/4th power of Gr_H is the ratio of the wall height to the wall shear layer thickness for a $Pr \ll 1$ fluid.

In the search of a consistent time scale-based physical interpretation of the Rayleigh and Boussinesq numbers, we consider the natural convection resulting flow with the *true* $v \sim (\alpha/H)(H/\delta_T)^2$ vertical velocity scale and we can find similar time ratios to that of Eq. (17) for a *natural convection Reynolds number*, $(Re_H)_{\mathrm{NC}}$. With the heat diffusing through the length H, a false diffusive length, and being convected with the velocity v through this same length H, one obtains

$$\frac{(\tau_{d,T})_H}{(\tau_c)_H} \sim \frac{H^2/\alpha}{H/[(\alpha/H)(H/\delta_T)^2]} \sim \left(\frac{\delta_T}{H}\right)^{-2} \tag{50}$$

Using the appropriate expression for the ratio δ_T/H [Eqs. (41) or (43)] one obtains

$$\frac{(\tau_{d,T})_H}{(\tau_c)_H} \sim \begin{cases} \sqrt{Ra_H} & \text{for } Pr \gg 1 \\ \sqrt{Bo_H} & \text{for } Pr \ll 1 \end{cases} \tag{51}$$

It can be concluded that the square roots of the Rayleigh and Boussinesq numbers are the relevant parameters for the heat transfer problem, which were unnecessarily squared. Comparing with Eq. (17), we can then conclude that $(Re_H)_{\mathrm{NC}} = \sqrt{Ra_H}$ for $Pr \gg 1$ and that $(Re_H)_{\mathrm{NC}} = \sqrt{Bo_H}$ for $Pr \ll 1$. It should be noted that $\sqrt{Bo_H}$ could be taken, more adequately, as a Péclet number, due to its independence on ν and its dependence on α. One concludes also that the transition to turbulence criterion remains established by the adequate Reynolds number, once again defined on a time scale basis, the transition occurring for $Re_H \sim 10^5 (Ra_H, Bo_H \sim 10^{10})$, even for very different physical situations.

Noting that the true diffusive length for heat is δ_T and not H, similarly to Eq. (18) we can obtain the time ratio

$$\frac{(\tau_{d,T})_{\delta_T}}{(\tau_c)_H} \sim \frac{\delta_T^2/\alpha}{H/[(\alpha/H)(H/\delta_T)^2]} \sim 1 \tag{52}$$

and give the following time scale-based interpretation to the thermal boundary layer thickness for the external natural convection situation: $\delta_T(H)$ is the distance, measured away from the solid wall, such that the time needed for the heat diffusion through the distance $\delta_T(H)$ is of the order of the time needed for convection from 0 to H, with the velocity $v(H) \sim (\alpha/H)(H/\delta_T)^2$.

The Rayleigh and Boussinesq numbers themselves could be obtained starting by using a false (but easy to calculate) velocity scale, that is, by considering $v \sim \alpha/H$. For $Pr \gg 1$, Eq. (32) expresses a momentum diffusion-buoyancy source balance within the thermal boundary layer, thus we take the following characteristic time ratio

$$\frac{(\tau_{d,v})_H}{(\tau_{b,T})_{v\sim\alpha/H}} \sim \frac{H^2/\nu}{(\alpha/H)/g\beta\Delta T} \sim Ra_H \tag{53}$$

For a $Pr \ll 1$ fluid, Eq. (32) is a momentum convection-buoyancy source balance within the thermal boundary layer, and the characteristic time ratio is now

$$\frac{(\tau_c)_H}{(\tau_{b,T})_{v\sim\alpha/H}} \sim \frac{H/(\alpha/H)}{(\alpha/H)/g\beta\Delta T} \sim Bo_H \tag{54}$$

The time-based physical meaning of the Rayleigh or the Boussinesq number, based on the height H and on the velocity scale $v \sim \alpha/H$, is the following: it is the ratio between the momentum diffusive or convective time scale and the buoyancy time needed to obtain a vertical velocity change $\Delta v \sim \alpha/H$. The use of the known wall height H to obtain the velocity scale $v \sim \alpha/H$ leads to easy calculations (only with known variables) and to an inappropriate (false) velocity scale, thus resulting in high values for Ra_H or Bo_H. The true v velocity scale for the thermal boundary layer is the one given by Eq. (36), and the adequate heat transfer governing dimensionless parameters are those present in Eq. (51).

6. External mixed convection

In this case, the fluid moves up under the combined influences of the imposed flow with velocity V_∞ and the buoyancy effect, leading to a combined forced and natural convection problem: a mixed convection problem.

The imposed upward vertical velocity is $v \sim V_\infty$ and the upward vertical buoyancy-induced velocity scales as $v \sim (\alpha/H)(H/\delta_T)^2$, as given by Eq. (36). The scale analysis proposed before treats the natural and forced convection *heat transfer problems*, respectively, as follows

$$\begin{aligned} &\text{NC: (convection)}_{(v)_{\text{NC}},H} \sim \text{(diffusion)}_{\alpha,(\delta_T)_{\text{NC}}} \\ &\text{FC: (convection)}_{(v)_{\text{FC}},H} \sim \text{(diffusion)}_{\alpha,(\delta_T)_{\text{FC}}} \end{aligned} \tag{55}$$

The forced or natural convection dominance for the heat transfer problem is obtained from the short time scale event, that is, using the convective terms on the left-hand side of Eq. (55),

$$\frac{(\tau_c)_{H,\text{FC}}}{(\tau_c)_{H,\text{NC}}} \sim \frac{H/(v)_{\text{FC}}}{H/(v)_{\text{NC}}} \sim \frac{(v)_{\text{NC}}}{(v)_{\text{FC}}} \begin{cases} < \text{O}(1), \\ \quad \text{Forced convection is dominant} \\ > \text{O}(1), \\ \quad \text{Natural convection is dominant} \end{cases} \tag{56}$$

where O(1) means *of the order of unit*.

If $Pr \gg 1$, Eq. (40) states that $(v)_{\text{NC}} \sim (\alpha/H) Ra_H^{1/2}$ and, as we have seen, $(v)_{\text{FC}} \sim V_\infty Pr^{-1/3}$ when analyzing the external forced convection thermal boundary layer for $Pr \gg 1$, the criterion (56) leading to

$$Pr \gg 1\text{:}\ \frac{Ra_H^{1/2} Pr^{1/3}}{Pe_H} \begin{cases} < \text{O}(1), \\ \quad \text{Forced convection is dominant} \\ > \text{O}(1), \\ \quad \text{Natural convection is dominant} \end{cases} \tag{57}$$

If $Pr \ll 1$, Eq. (42) states that $(v)_{\text{NC}} \sim (\alpha/H) Bo_H^{1/2}$ and, as we have seen, $(v)_{\text{FC}} \sim V_\infty$ when analyzing the external forced convection thermal boundary layer for $Pr \ll 1$, the criterion (56) leads to

$$Pr \ll 1\text{:}\ \frac{Bo_H^{1/2}}{Pe_H} \begin{cases} < \text{O}(1), \\ \quad \text{Forced convection is dominant} \\ > \text{O}(1), \\ \quad \text{Natural convection is dominant} \end{cases} \tag{58}$$

The ratio Gr_H/Re_H^2, sometimes referred to as the Richardson number [3], is the largely used parameter to characterize the mixed convection situation for any Prandtl number, which is usually obtained from the nondimensionalized momentum only, without the true vertical velocity scales. Such a parameter can also be obtained as

$$\frac{(\tau_c)_H]_{\text{FC}}}{\tau_b} \sim \frac{H/V_\infty}{V_\infty/g\beta\Delta T} \sim \frac{Gr_H}{Re_H^2} \tag{59}$$

that is, using the $v \sim V_\infty$ velocity scale for both the buoyancy time scale and the convective time scale. It should be noted that the particular ratio $Bo_H^{1/2}/Pe_H$ of criterion (58) can be expressed as the square root of the ratio Gr_H/Re_H^2.

From Eqs. (57) and (58) one concludes that the ratio Gr_H/Re_H^2 is not the suitable parameter to characterize the $Pr \gg 1$ mixed convection situation, and that the square root of the ratio Gr_H/Re_H^2 is the suitable parameter to characterize the $Pr \ll 1$ situation. If we recall that $\sqrt{Ra_H}$ or $\sqrt{Bo_H}$ are the suitable parameters to characterize the natural convection situations for $Pr \gg 1$ or $Pr \ll 1$, respectively, and not the Ra_H or Bo_H parameters, this square root is not a strange result.

Criterion (56) is convection time scale-based (or velocity based). Bejan [1], starting from the idea that the (forced or natural) thinner thermal boundary layer rules the heat transfer mechanism from the wall to the fluid, states a boundary layer thickness-based criterion. The ruling parameters obtained with this criterion are the square roots of the ones obtained with the here proposed time scale-based criterion, the square roots appearing because Eq. (55) gives the diffusion time (and thus also the convective time) scaling with δ_T^2/α. Bejan [1] presents a discussion about the suitable parameters to characterize the heat transfer problem in mixed convection, with similar conclusions.

7. Bénard convection

Another situation, physically very different from the foregoing ones, that can be easily analyzed with the proposed time scale based methodology is the Bénard convection problem. When ΔT is lower than its critical value, the fluid is quiescent and thermally stratified (Fig. 5(a)), the heat transfer from the bottom to the top wall occurring by pure diffusion through the fluid. If ΔT reaches its critical value, the fluid starts to move in counterrotating two-dimensional (almost square) rolls, as shown in Fig. 5(b), the heat transfer between walls occurring as a combination of diffusion and convection: it is the onset of convection.

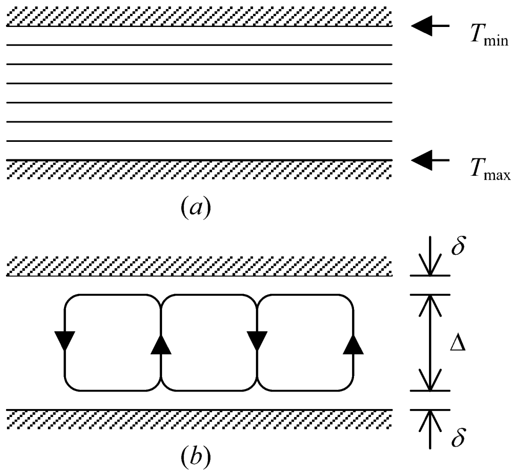

Fig. 5. Bénard convection: (a) Stagnant stratified fluid; (b) The onset of convection, with (almost) square counterrotating rolls.

In the first situation, Fig. 5(a), heat is transferred by pure diffusion across all the layer height H. When the onset of convection occurs, for a $Pr \sim 1$ fluid, heat is transferred by diffusion across the lower and upper layers of thickness δ adjacent to the walls, and by convection in the remaining path fraction, Δ. Comparatively with the pure diffusion situation, the main difference at the onset of convection exists only at the interior Δ layer.

Without convection, the diffusive time across the Δ layer scales as

$$(\tau_{d,T})_\Delta \sim \frac{\Delta^2}{\alpha} \tag{60}$$

When the fluid motion begins, the convective time scale across this same Δ layer is

$$(\tau_c)_\Delta \sim \frac{\Delta}{v} \tag{61}$$

The v velocity scale is evaluated from the momentum balance for a single half square roll, a way followed also by Bejan [6]. The buoyancy force scales as $\rho(\Delta^2/2)g\beta(\Delta T/n)$, where $\Delta T/n$ is the scale of the temperature difference between the rising fluid and the average temperature of the fluid within the layer. The buoyancy force is balanced by the shear force scaling as $2[(\rho\nu v/\delta)\Delta/2]$, the term within the curved parenthesis being the scale for the viscous shear stress. The convective velocity scales as $v \sim (1/2n)(g\beta\Delta T H^2/\nu)(\Delta/H)(\delta/H)$, and the convective time scale across the Δ layer becomes

$$(\tau_c)_\Delta \sim 2n\left(\frac{g\beta\Delta T H}{\nu}\right)^{-1}\left(\frac{\delta}{H}\right)^{-1} \tag{62}$$

The onset of convection occurs when the convective time scale becomes shorter than the diffusive time scale, the criterion for the onset of convection being $(\tau_c)_\Delta/(\tau_{d,T})_\Delta \sim 1$, that is,

$$Ra_H \sim 2n\left[\frac{\delta}{H}\left(1-\frac{2\delta}{H}\right)^2\right]^{-1} \tag{63}$$

invoking the geometric argument $\Delta = H - 2\delta$. The Rayleigh number Ra_H is defined as in Eq. (39), noting that H is now the global layer thickness of fluid and ΔT is the temperature diference between the lower and upper horizontal walls. As a *reasonable* approach, in a scale sense, we can consider that $n = 4$ (the temperature difference scales as $\Delta T/4$ in each δ layer, and as $\Delta T/2$ in the interior Δ layer), and that $\delta/H \sim 1/4$, thus obtaining that the onset of convection occurs for $Ra_H \sim 128$, an O(100) value.

The *true* critical value for this governing dimensionless parameter is well established as 1708, the here obtained value being one order of magnitude lower, but considerably greater than 1. This discrepancy is due mainly to the geometric factors of order 1, that combined describes only poorly the geometric characteristics of the rolls filling the enclosed space [6], and to the pressure forces that were not considered: the resulting flow is not a boundary layer flow but a recirculating flow. However, the essential of the involved phenomena has been retained and the correct governing dimensional parameter has been obtained by the time scale based analysis proposed.

One can go one step further on the scale analysis of the Bénard convection problem by noting that we are comparing the diffusive and convective time scales without the consideration of the effective areas *used* for the diffusive and convective transport phenomena. On the onset of convection, due to the emerging flow structure, the convective heat transfer occurs across a fraction of the total width only. For a single roll we can assume, as a first approach, that the horizontal length entering in the area used for convection heat transfer from the lower to the upper boundary is only $\Delta/4$, instead of the full Δ length used for the conduction heat transfer before the onset of convection. For a better comparison of the time scales only one should take the same reference area as used for convection and diffusion, because the essential of the problem is the resulting heat flow. Taking the full Δ length as reference, the v' velocity scale for time scale comparison is obtained from $v'\Delta \sim v(\Delta/4)$ as $v' \sim v/4$. With the consideration of these additional geometric features, in the form of geometric parameters of O(1), one would obtain that the critical Rayleigh number for the onset of convection is $Ra_H \sim 512$, a much better result taking present the well-known value of 1708.

Without the introduction of the geometric factors, one would obtain the criterion for the onset of convection stating that it occurs for $Ra_H \sim 1$, a much poor numerical result. In fact, when the convection onsets, the diffusion is not occurring through the H height, as well as the convection is not occurring through this same distance. The introduced geometric factors appear and act as *better* approaches for the *real* occurring phenomena.

Once again, even for a non-boundary layer laminar situation, the time scale based approach proposed shows to be a very effective and attractive tool for analysis and evaluation of the correct dimensionless parameters and transition's criteria.

8. Conclusions

The time scale-based methodology proposed in the present paper proves to be a powerful and unifying tool for momentum and thermal boundary layer analysis. The time scales for the different individual phenomena are easily obtained, and the usual differential equations can be interpreted as algebraic relationships between different time scales for convection, diffusion and source. It is a method with its own physical insight, any situation being dominated by the short time scale events present.

The method is very effective when applied to the studied situations, to obtain the momentum and thermal boundary layer thickness, for the definition of the adequate governing dimensionless parameters, and for a unified, consistent and physically coherent interpretation of these parameters. Even the numerical values usually assumed by such parameters can be understood with this time scale basis.

The criterion of transition to turbulence is established through the Reynolds number, interpreted as a diffusion-convection time ratio, with $Re_{\text{transition}} \sim \tau_{\text{diffusion}}/\tau_{\text{convection}} \sim 10^5$ for very different physical situations. The unifying Reynolds number is obtained taking the external boundary layer flow situation as reference, thus measuring the ratio between the time needed for the wall information diffusion (normal to the wall) and the time needed for the convective transfer along the corresponding length, under the prevailing velocity. Such results suggest that the fundamental mechanism of transition is the same in apparently very different physical situations, and that they are not so different if they are analyzed on a common basis. The similarity between the unified results proposed for transition, obtained in the form of a Reynolds number as a diffusion-convection time ratio, and these proposed by Bejan [1,7], should be noted. The Bejan transition analysis [7] starts with the very small flow perturbations, known as the infinitesimal flow buckling, to obtain the common sinusoidal character of the first disturbances that result in the transition to turbulence. The transition Reynolds number, based on the adequate transverse length, is of the order of 100, which is defined as the diffusion-buckling time scale ratio. In the present analysis, any boundary layer problem is analyzed taking the external boundary layer flow as reference, as well as the corresponding transition Reynolds number.

Also the Bénard convection problem is analyzed with this time scale based approach, in the laminar regime, the criterion for the onset of convection being also found as a convection–diffusion time ratio. If it is true that numerical values should be absent, in principle, in any scale analysis, it is also true that it is tempting to introduce some *reasonable* factors in order to obtain better numerical results and conclusions. This is the case of the preceding internal flow and Bénard convection scale analysis.

The constructal theory proposed by Bejan [6] establishes that an open system evolves in well defined ways, such that the access to the imposed (global) currents that flow through it is easiest. In the time scale based approach proposed, the system has the freedom to choose between diffusion or convection as the governing transfer mechanisms for heat and momentum transfer, the easier mechanism being the one associated with the short time scale. This is the most global message of Eq. (8), which remains unchangeable if we are thinking about a 3D problem.

The time scale based analysis proposed is a methodology with its own physical insight, leading to very good results when applied to the analyzed situations. In the future, it needs to be used more and more in order to evaluate its relative merits and demerits by comparison with the classical use of the scale analysis method, applied over the primitive differential equations.

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